Stationary motion of active Brownian particles

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The stationary motion of active Brownian particles is studied by using the stochastic averaging method for quasi-integrable Hamiltonian systems. First the stochastic averaging method for quasi-integrable Hamiltonian systems is briefly introduced. Then the stationary solution of the dynamic equations governing an active Brown particle in plane with the Rayleigh velocity-dependent friction model subject to Gaussian white noise excitations is obtained by using the stochastic averaging method. The solution is validated by comparison with the result from Monte Carlo simulation. Finally, two more stationary solutions of the dynamic equations governing active Brownian particle with the Schienbein-Gruler and Erdmann velocity-dependent friction models, respectively, subject to Gaussian white noise excitations are also given.

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I. INTRODUCTION

Active Brownian particles are Brownian particles with the ability to store energy which can be used for active movement. In the last decade, the theory of active Brownian particles has been developed rapidly [1–6] due to its potential application to collective movement in biological and social swarms. For example, self-driven motion of particles can be observed in physicochemical systems [7]. In biological systems, ranging from cells [8,9] to higher organisms, such as birds [10,11], self-driven motion can also be found. Even human movement [12] and the movement to traffic systems [13] can also be described as active motion.

The motion of active Brownian particles is usually described by using Langevin equations, which can be modeled as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equations by adding Wong-Zakai correction terms. Usually, these equations cannot be solved analytically unless they are linear. So, instead of solving these equations, the associated Fokker-Planck equation is solved. Recently, Ebeling *et al.* [14,15] proposed a theory of canonical-dissipative systems and applied it to active Brownian particles. The exact stationary solution of the Fokker-Planck equation was obtained for some special cases. However, canonical-dissipative systems are only a subclass of the so-called stochastically excited and dissipated Hamiltonian systems, for which a whole theoretical framework has been established in the field of mechanics by one of the present authors (W.Q.Z.) and his co-workers in the last decade [16]. The theoretical framework includes three procedures for predicting the response of the systems, i.e., the exact stationary solution [17–19], the equivalent nonlinear system method [20–22] and the stochastic averaging method for quasi-Hamiltonian systems [23–25]. The basic idea of these procedures is that the functional form of the solution of a stochastically excited and dissipated Hamiltonian system depends upon the integrability and resonance of its associated Hamiltonian system. Five classes of solutions were obtained by using these procedures for five groups of the systems: nonintegrable, integrable and nonresonant, integrable and resonant, partially integrable and nonresonant, and partially integrable and resonant.

As an application of the theory of stochastically excited and dissipated Hamiltonian systems in the dynamics of active Brownian particles, in this paper, the stationary behavior of a Brownian particle in plane with Rayleigh, Schienbein-Gruler, and Erdmann velocity-dependent friction models, respectively, under Gaussian white noise excitations is studied by using the stochastic averaging method for quasi-integrable Hamiltonian systems. First, the concept of quasi-integrable Hamiltonian systems, the stochastic averaging method for them, and the stationary solution of the averaged Fokker-Planck equation are introduced in Secs. II–IV, respectively. Then the stationary solution of the dynamical equations governing an active Brownian particle with Rayleigh velocity-dependent friction model subject to Gaussian white noise excitations is obtained by using the stochastic averaging method and verified by comparison with the result from Monte Carlo simulation in Sec. V. The stationary solutions for the dynamical equations describing an active Brown particle with Schienbein-Gruler and Erdmann velocity-dependent friction models are given in Sec. VI. It is shown that the stochastic averaging method yields quite good analytical solution.

II. QUASI-INTEGRABLE HAMILTONIAN SYSTEMS

An n degree-of-freedom Hamiltonian dynamical system is governed by the following n pairs of Hamilton equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1,2, \ldots ,n, \tag{1}$$

where q_i and p_i are generalized displacements and generalized momenta, respectively; $H=H(\boldsymbol{q},\boldsymbol{p})$ is a Hamiltonian

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with continuous first-order derivatives. A Hamiltonian system of n degrees of freedom is said to be integrable or completely integrable if there exist n independent integrals of the motion $H_1=H, H_2, \ldots, H_n$ which are in involution. This last term means that the Poisson bracket of any two integrals of motion H_i and H_j vanishes, i.e.,

$$[H_i, H_j] = \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} = 0; \quad i,j,k = 1,2,\ldots,n. \tag{2}$$

A quasi-Hamiltonian system of n degrees of freedom is governed by the following equations of motion:

$$\frac{dq_i}{dt} = \frac{\partial H'}{\partial p_i}, \tag{3a}$$

$$\frac{dp_i}{dt} = -\frac{\partial H'}{\partial q_i} - \varepsilon c_{ij}\frac{\partial H'}{\partial p_j} + \varepsilon^{1/2} f_{ik}\xi_k(t),$$

$$i = 1,2,\ldots,n, \quad k = 1,2,\ldots,m, \tag{3b}$$

where $H'=H'(\boldsymbol{q},\boldsymbol{p})$ is a twice differentiable Hamiltonian; $c_{ij}=c_{ij}(\boldsymbol{q},\boldsymbol{p})$ are differentiable functions; $f_{ik}=f_{ik}(\boldsymbol{q},\boldsymbol{p})$ are twice differentiable functions; ε is a small positive parameter; $\xi_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions

$$E[\xi_k(t)\xi_l(t+\tau)] = 2D_{kl}\delta(\tau), \quad k,l = 1,2,\ldots,m. \tag{4}$$

The second summation terms on the right-hand side of Eq. (3b) may represent a set of lightly linear and (or) nonlinear friction forces while the third summation terms may include weakly external and (or) parametric excitations of Gaussian white noise. Equations (3a) and (3b) can be modeled as Stratonovich stochastic differential equations and then converted into the following Itô stochastic differential equations:

$$dq_i = \frac{\partial H'}{\partial p_i}dt, \tag{5a}$$

$$dp_i = -\left(\frac{\partial H'}{\partial q_i} + \varepsilon c_{ij}\frac{\partial H'}{\partial p_j} - \varepsilon D_{kl} f_{jl}\frac{\partial f_{ik}}{\partial p_J}\right)dt + \varepsilon^{1/2}\sigma_{ik}dB_k(t),$$

$$i,j = 1,2,\ldots,n, \quad k,l = 1,2,\ldots,m, \tag{5b}$$

where $B_k(t)$ are the standard Wiener processes and $\boldsymbol{\sigma\sigma}^T = 2\,\mathbf{fDf}^T$. The third summation terms on the right-hand side of Eq. (5b) are known as the Wong-Zakai correction terms [26]. These terms can usually be split into two parts: one has the effect of modifying the conservative forces and the other of modifying the friction forces. The first part can be combined with $\partial H'/\partial q_i$ to form overall effective conservative forces $\partial H/\partial q_i$ with a modified Hamiltonian $H=H(\boldsymbol{q},\boldsymbol{p})$and $\partial H/\partial p_i = \partial H'/\partial p_i$. The second part may be combined with $\varepsilon c_{ij}\partial H'/\partial p_j$ to constitute effective friction forces $\varepsilon m_{ij}\partial H/\partial p_j$ with $m_{ij}=m_{ij}(\boldsymbol{q},\boldsymbol{p})$. With these accomplished, Eqs. (5a) and (5b) can be rewritten as

$$dq_i = \frac{\partial H}{\partial p_i}dt, \tag{6a}$$

$$dp_i = -\left(\frac{\partial H}{\partial q_i} + \varepsilon m_{ij}\frac{\partial H}{\partial p_j}\right)dt + \varepsilon^{1/2}\sigma_{ik}dB_k(t),$$

$$i,j = 1,2,\ldots,n, \quad k,l = 1,2,\ldots,m. \tag{6b}$$

In the following it is assumed that the Hamiltonian system governed by Eqs. (6a) and (6b) with $\varepsilon=0$ is integrable. Then, Eqs. (6a) and (6b) describe a quasi-integrable Hamiltonian system.

III. STOCHASTIC AVERAGING METHOD FOR QUASI-INTEGRABLE HAMILTONIAN SYSTEMS

Consider the quasi-integrable Hamiltonian system governed by Eqs. (6a) and (6b). Introduce the transformation

$$H_r = H_r(\boldsymbol{q},\boldsymbol{p}), \quad \Theta_r = \Theta_r(\boldsymbol{q},\boldsymbol{p}), \quad r = 1,2,\ldots,n, \tag{7}$$

where Θ_r are angle variables. The Itô equations for H_r and Θ_r are obtained from Eqs. (6a) and (6b) by using transformation (7) and the Itô differential rule [27] as follows:

$$dH_r = \varepsilon\left(-m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i} + D_{kl} f_{ik} f_{jl}\frac{\partial^2 H_r}{\partial p_i\,\partial p_j}\right)dt$$

$$+ \varepsilon^{1/2}\frac{\partial H_r}{\partial p_i}\sigma_{ik}dB_k(t), \tag{8a}$$

$$d\Theta_r = \left(\omega_r - \varepsilon m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial \Theta_r}{\partial p_i} + \varepsilon D_{kl} f_{ik} f_{jl}\frac{\partial^2 \Theta_r}{\partial p_i\,\partial p_j}\right)dt$$

$$+ \varepsilon^{1/2}\frac{\partial \Theta_r}{\partial p_i}\sigma_{ik}dB_k(t), \tag{8b}$$

where q_i and p_i on the right-hand side of Eqs. (8a) and (8b) should be replaced by H_r and Θ_r in terms of Eq. (7). The form and dimension of the stochastic averaging equations of a quasi-integrable Hamiltonian system depend on whether the associated Hamiltonian system is resonant or not.

In the nonresonant case $\Theta_1, \Theta_2, \ldots, \Theta_n$ in Eq. (8b) are rapidly varying processes while $H_1, H_2, \ldots, H_n$ in Eq. (8a) are slowly varying ones. According to a theorem due to Khasminskii [28], the H_r converge weakly to an n-dimensional diffusion process as $\varepsilon \to 0$ in a time interval $0 \leqslant t \leqslant T$, where $T \sim 0(\varepsilon^{-1})$. In other words, the H_r may be replaced in the first approximation by an n-dimensional diffusion process for small ε. For simplicity, the same symbol H_r will be used to denote the rth component of this diffusion process.

The averaged Itô equations for this n-dimensional diffusion process are obtained by applying time averaging to Eq. (8a) under the condition that the H_r on the right-hand side of Eq. (8a) is kept constant. The time averaging can be replaced by phase space averaging over $\Theta_r(r=1,2,\ldots,n)$ since the motion of the associated Hamiltonian system on the constant $H_r(r=1,2,\ldots,n)$ surface is ergodic. Thus, the averaged Itô equations for H_r are of the form

$$dH_r = \varepsilon U_r(\boldsymbol{H})dt + \varepsilon^{1/2}V_{rk}(\boldsymbol{H})dB_k(t), \quad r=1,2,\ldots,n, \quad k=1,2,\ldots,m, \tag{9}$$

and the averaged Fokker-Planck equation associated with Eq. (9) is

$$\frac{\partial p}{\partial t} = \varepsilon\left\{-\frac{\partial}{\partial H_r}[a_r(\boldsymbol{H})p] + \frac{1}{2}\frac{\partial^2}{\partial H_r\,\partial H_s}[b_{rs}(\mathbf{H})p]\right\} \tag{10}$$

where $\boldsymbol{H}=[H_1,H_2,\ldots,H_n]^T$,

$$a_r(\boldsymbol{H}) = U_r(\boldsymbol{H}) = \left\langle -m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i} + D_{kl}f_{ik}f_{jl}\frac{\partial^2 H_r}{\partial p_i\,\partial p_j}\right\rangle, \tag{11a}$$

$$b_{rs}(\boldsymbol{H}) = V_{rk}V_{sk} = \left\langle 2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial H_s}{\partial p_j}\right\rangle,$$

$$r,s,i,j=1,2,\ldots,n, \quad k,l=1,2,\ldots,m, \tag{11b}$$

and

$$\langle\cdot\rangle = \frac{1}{(2\pi)^n}\int_0^{2\pi}(\cdot)d\theta_1 d\theta_2\cdots d\theta_n \tag{12}$$

denotes an averaging operator. In Eq. (10), $p=p(\boldsymbol{H},t|\boldsymbol{H}_0)$ with initial condition

$$p(\boldsymbol{H},0|\boldsymbol{H}_0) = \delta(\boldsymbol{H}-\boldsymbol{H}_0) \tag{13}$$

or $p=p(\boldsymbol{H},t)$ with initial condition

$$p(\boldsymbol{H},0) = p(\boldsymbol{H}_0), \tag{14}$$

depending upon whether an initial state or an initial probability density is specified. The Fokker-Planck equation (10) is also subjected to appropriate boundary conditions,

$$-a_r(\boldsymbol{H})p + \frac{1}{2}\frac{\partial}{\partial H_s}[b_{rs}(\boldsymbol{H})p] = 0, \quad r,s=1,2,\ldots,n, \quad \boldsymbol{H}\in S, \tag{15}$$

which imply vanishing probability flows in n directions at the boundary.

It is resonant case if there exist the following $\alpha(1\leqslant\alpha\leqslant n-1)$ resonant relations:

$$k_r^u\omega_r = 0(\varepsilon), \quad u=1,2,\ldots,\alpha, \quad r=1,2,\ldots,n. \tag{16}$$

Introduce α combinations ϕ_u or angle variables

$$\phi_u = k_r^u\Theta_r, \quad u=1,2,\ldots,\alpha, \quad r=1,2,\ldots,n. \tag{17}$$

The Itô equations for ϕ_u can be obtained through appropriate combinations of Eq. (8b) as follows:

$$d\phi_u = \left(0(\varepsilon) - \varepsilon m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial\phi_u}{\partial p_i} + \varepsilon D_{kl}f_{ik}f_{jl}\frac{\partial^2\phi_u}{\partial p_i\,\partial p_j}\right)dt + \frac{\partial\phi_u}{\partial p_i}\sigma_{ik}dB_k(t). \tag{18}$$

It is seen from Eqs. (8a), (8b), and (18) that $H_1,H_2,\ldots,H_n$ and $\phi_1,\phi_2,\ldots,\phi_a$ are slowly varying process while $\Theta_1,\Theta_2,\ldots,\Theta_{n-a}$ are rapidly varying process. Based on the Khasminskii theorem [28], the H_r and ϕ_u converge to an $(n+a)$-dimensional diffusion process as $\varepsilon\to 0$ in a time interval $0\leqslant t\leqslant T$, where $T\sim 0(\varepsilon^{-1})$.

The Itô equations for this $(n+a)$-dimensional diffusion process are obtained by applying time averaging to Eqs. (8a) and (18) under the condition that H_r,ϕ_u on the right-hand sides are regarded as constant. For a similar reason as in the nonresonant case, this time averaging can be replaced by phase space averaging with respect to $\Theta_1,\Theta_2,\ldots,\Theta_{n-a}$. Thus, we obtain the following averaged Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \varepsilon\left\{-\frac{\partial}{\partial H_r}[a_r(\boldsymbol{H},\boldsymbol{\Phi})p] - \frac{\partial}{\partial\phi_u}[a_u(\boldsymbol{H},\boldsymbol{\Phi})p] + \frac{1}{2}\frac{\partial^2}{\partial H_r\,\partial H_s}[b_{rs}(\boldsymbol{H},\boldsymbol{\Phi})p] + \frac{1}{2}\frac{\partial^2}{\partial H_r\,\partial\phi_u}[b_{ru}(\boldsymbol{H},\boldsymbol{\Phi})p] + \frac{1}{2}\frac{\partial^2}{\partial\phi_u\,\partial H_r}[b_{ur}(\boldsymbol{H},\boldsymbol{\Phi})p] + \frac{1}{2}\frac{\partial^2}{\partial\phi_u\,\partial\phi_v}[b_{uv}(\boldsymbol{H},\boldsymbol{\Phi})p]\right\} \tag{19}$$

where $\boldsymbol{\Phi}=[\phi_1,\phi_2,\ldots,\phi_a]^T$,

$$a_r = \left\langle -m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial H_r}{\partial p_i} + D_{kl}f_{ik}f_{jl}\frac{\partial^2 H_r}{\partial p_i\,\partial p_j}\right\rangle,$$

$$b_{rs} = \left\langle 2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial H_s}{\partial p_j}\right\rangle,$$

$$a_u = \left\langle 0(\varepsilon) - m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial\phi_u}{\partial p_i} + D_{kl}f_{ik}f_{jl}\frac{\partial^2\phi_u}{\partial p_i\,\partial p_j}\right\rangle,$$

$$b_{ru} = \left\langle 2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial\phi_u}{\partial p_j}\right\rangle, \tag{20}$$

$$b_{uv} = \left\langle 2D_{kl}f_{ik}f_{jl}\frac{\partial\phi_u}{\partial p_i}\frac{\partial\phi_v}{\partial p_j}\right\rangle, \quad r,s,i,j=1,2,\ldots,n,$$

$$u,v=1,2,\ldots,a, \quad k,l=1,2,\ldots,m,$$

and

$$\langle\cdot\rangle = \frac{1}{(2\pi)^{n-a}}\int_0^{2\pi}(\cdot)d\theta_1 d\theta_2\cdots d\theta_{n-a} \tag{21}$$

denotes an averaging operator. In Eq. (19), $p=p(\boldsymbol{H},\boldsymbol{\Phi},t|\boldsymbol{H}_0,\boldsymbol{\Phi}_0)$ with initial condition

$$p(\boldsymbol{H},\boldsymbol{\Phi},0|\boldsymbol{H}_0,\boldsymbol{\Phi}_0) = \delta(\boldsymbol{H}-\boldsymbol{H}_0)\delta(\boldsymbol{\Phi}-\boldsymbol{\Phi}_0) \tag{22}$$

or $p=p(\boldsymbol{H},\boldsymbol{\Phi},t)$ with initial condition

$$p(\boldsymbol{H},\boldsymbol{\Phi},0) = p(\boldsymbol{H}_0,\boldsymbol{\Phi}_0). \tag{23}$$

The averaged Fokker-Planck equation (19) is also subjected to appropriate boundary conditions. For example, the conditions at the boundary $\boldsymbol{H}\in S$ are similar to those in Eq. (15)

while those with respect to ϕ_u are that p is a periodic function of ϕ_v with period 2π.

IV. STATIONARY SOLUTIONS OF AVERAGED FOKKER-PLANCK EQUATIONS

One advantage of the stochastic averaging method for quasi-integrable Hamiltonian systems introduced in the last section is reducing the dimension of the equations from $2n$ to n or $n+a<2n$. Another advantage of the stochastic averaging method is simplifying the equations such that in the averaged Fokker-Planck equation there is only potential probability flow and no circulatory probability flow. Under boundary condition (15), the probability flow vanishes everywhere inside the boundary, that is, the averaged system belongs to the class of stationary potentials and the exact stationary solution of the averaged Fokker-Planck equation, if it exists, can be obtained easily.

The stationary Fokker-Planck equation (10) with vanishing time-derivative term under boundary condition (15) can be further reduced to the following n equations for vanishing probability potential flow in n directions:

$$-a_r(\boldsymbol{H})p+\frac{1}{2}\frac{\partial}{\partial H_s}[b_{rs}(\boldsymbol{H})p]=0,\quad r,s=1,2,\ldots,n. \tag{24}$$

The solution of Eq. (24) is of the form

$$p(\boldsymbol{H})=C\exp[-\lambda(\boldsymbol{H})], \tag{25}$$

where $\lambda(\boldsymbol{H})$ is the so-called probability potential and C is a normalization constant. Substituting Eq. (25) into Eq. (24), one obtains

$$b_{rs}\frac{\partial\lambda}{\partial H_s}=\frac{\partial b_{rs}}{\partial H_s}-2a_r,\quad r,s=1,2,\ldots,n. \tag{26}$$

These are n first-order linear partial differential equations for λ as a function of H_r. If the matrix $\boldsymbol{B}=[b_{rs}]$ is nonsingular, so that its inverse $\boldsymbol{B}^{-1}=\boldsymbol{G}=[g_{rs}]$ exists, then Eq. (26) may be simplified to

$$\frac{\partial\lambda}{\partial H_s}=g_{ir}\left(\frac{\partial b_{rs}}{\partial H_s}-2a_r\right). \tag{27}$$

If the following compatibility conditions are satisfied:

$$\frac{\partial\lambda}{\partial H_i\,\partial H_j}=\frac{\partial\lambda}{\partial H_j\,\partial H_i},\quad i,j=1,2,\ldots,n, \tag{28}$$

then a consistent λ function can be obtained as follows:

$$\lambda(\boldsymbol{H})=\int_0^{H_1}\frac{\partial\lambda}{\partial H_1}dH_1+\int_0^{H_2}\frac{\partial\lambda}{\partial H_2}dH_2+\cdots+\int_0^{H_n}\frac{\partial\lambda}{\partial H_n}dH_n. \tag{29}$$

The exact stationary solution $p(\boldsymbol{H})$ of the averaged Fokker-Planck Eq. (10) is then obtained by substituting Eq. (29) into Eq. (25). The stationary probability density of generalized displacements and momenta can be derived from Eq. (25) as follows:

$$p(\boldsymbol{q},\boldsymbol{p})=p(\boldsymbol{H},\boldsymbol{\theta})\left|\frac{\partial(\boldsymbol{H},\boldsymbol{\theta})}{\partial(\boldsymbol{q},\boldsymbol{p})}\right|=p(\boldsymbol{\theta}|\boldsymbol{H})p(\boldsymbol{H})\left|\frac{\partial(\boldsymbol{H},\boldsymbol{\theta})}{\partial(\boldsymbol{q},\boldsymbol{p})}\right| =C_1p(\boldsymbol{H}), \tag{30}$$

where $|\partial(\boldsymbol{H},\boldsymbol{\theta})/\partial(\boldsymbol{q},\boldsymbol{p})|$ is the absolute value of the Jacobian determinant of the transformation (7) and usually equal to a constant. The stationary statistics, such as the marginal probability density, mean value, and mean-square value of generalized displacements and momenta, can be obtained from Eq. (30) by definitions.

The exact stationary solution of the averaged Fokker-Planck equation (19) is slightly more difficult to obtain [24]. Here one way suitable to the following application is given. Obviously, if p satisfies the following $n+a$ first-order partial differential equations, then it will be the stationary solution of the averaged Fokker-Planck equation (19):

$$-a_r(\boldsymbol{H},\boldsymbol{\Phi})p+\frac{1}{2}\frac{\partial}{\partial H_s}[b_{rs}(\boldsymbol{H},\boldsymbol{\Phi})p]+\frac{1}{2}\frac{\partial}{\partial\phi_u}[b_{ru}(\boldsymbol{H},\boldsymbol{\Phi})p]=0, \tag{31}$$

$$-a_u(\boldsymbol{H},\boldsymbol{\Phi})p+\frac{1}{2}\frac{\partial}{\partial H_r}[b_{ur}(\boldsymbol{H},\boldsymbol{\Phi})p]+\frac{1}{2}\frac{\partial}{\partial\phi_v}[b_{uv}(\boldsymbol{H},\boldsymbol{\Phi})p]=0,$$

$$r,s=1,2,\ldots,n,\quad u,v=1,2,\ldots,\alpha.$$

Based on the boundary conditions in Eq. (15), the stationary solution of Eq. (19) is assumed of the following form:

$$p(\boldsymbol{H},\boldsymbol{\Phi})=C\exp[-\lambda(\boldsymbol{H},\boldsymbol{\Phi})]. \tag{32}$$

Substituting Eq. (32) into Eq. (31), one obtains

$$b_{rs}\frac{\partial\lambda}{\partial H_s}+b_{ru}\frac{\partial\lambda}{\partial\phi_u}=\frac{\partial b_{rs}}{\partial H_s}+\frac{\partial b_{ru}}{\partial\phi_u}-2a_r,$$

$$b_{ur}\frac{\partial\lambda}{\partial H_r}+b_{uv}\frac{\partial\lambda}{\partial\phi_v}=\frac{\partial b_{ur}}{\partial H_r}+\frac{\partial b_{uv}}{\partial\phi_v}-2a_u. \tag{33}$$

If the following compatibility conditions are satisfied:

$$\frac{\partial\lambda}{\partial H_r\,\partial H_s}=\frac{\partial\lambda}{\partial H_s\,\partial H_r},\quad\frac{\partial\lambda}{\partial H_r\,\partial\phi_v}=\frac{\partial\lambda}{\partial\phi_v\,\partial H_r},$$

$$\frac{\partial\lambda}{\partial\phi_u\,\partial\phi_v}=\frac{\partial\lambda}{\partial\phi_v\,\partial\phi_u}, \tag{34}$$

then a consistent λ function can be obtained as follows:

$$\lambda(\boldsymbol{H},\boldsymbol{\Phi})=\int_0^{H_r}\frac{\partial\lambda}{\partial H_r}dH_r+\int_0^{\phi_u}\frac{\partial\lambda}{\partial\phi_u}\,\partial\phi_u,$$

$$r=1,2,\ldots,n,\quad u=1,2,\ldots,\alpha. \tag{35}$$

The stationary probability density of generalized displacements and momenta can be derived from Eq. (32) as follows:

$$p(\boldsymbol{q},\boldsymbol{p})=C_2p(\boldsymbol{H},\boldsymbol{\Phi}). \tag{36}$$

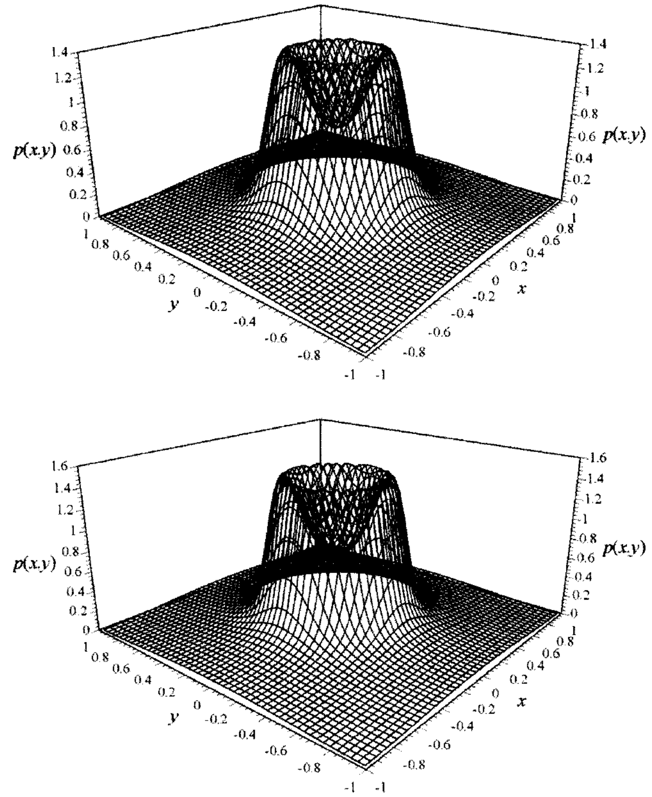

FIG. 1. Stationary probability density $p(x,y)$ of displacements x,y of system (37) with Rayleigh friction model in the case far from equilibrium (top) from simulation of Eq. (37) and (bottom) from the analytical expression in Eq. (55). The parameters are $\gamma_1=4$, $\gamma_2=1$, $\omega=6$, and $D=1$.

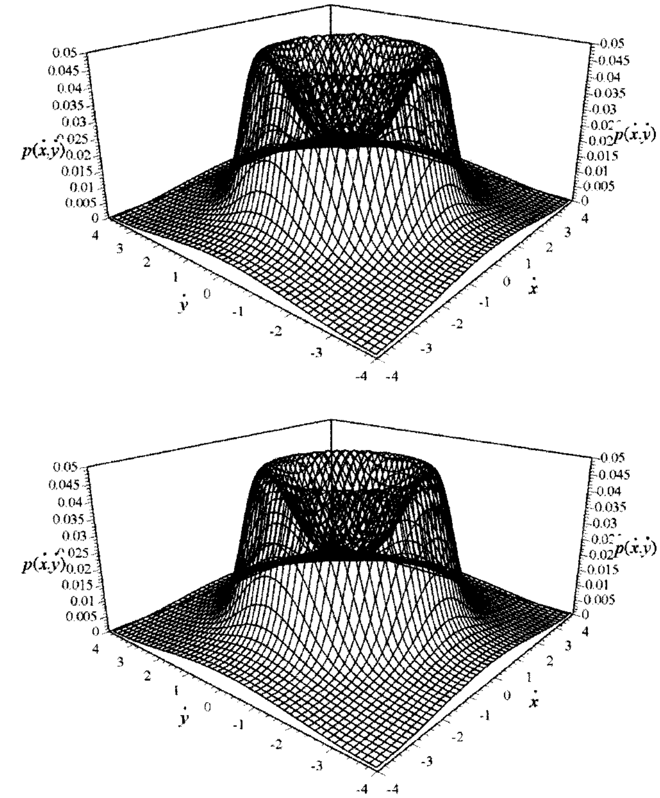

FIG. 2. Stationary probability density $p(\dot{x},\dot{y})$ of velocities $\dot{x},\dot{y}$ of system (37) with Rayleigh friction model in the case far from equilibrium (top) from simulation of Eq. (37) and (bottom) from the analytical expression in Eq. (55). The parameters are the same as those in Fig. 1.

V. STATIONARY SOLUTION FOR RAYLEIGH FRICTION MODEL

The motion of a Brownian particle with unit mass $m=1$, position x,y, and velocity $\dot{x},\dot{y}$ moving in a two-dimensional parabolic potential subjected to Gaussian white noise excitations is governed by the following equations:

$$\ddot{x} + F\dot{x} + \omega^2 x = \xi_x(t), \quad \ddot{y} + F\dot{y} + \omega^2 y = \xi_y(t), \tag{37}$$

where ω is the oscillation frequency, $F=F(x,y,\dot{x},\dot{y})$ is the coefficient of active friction, and $\xi_x(t)$ and $\xi_y(t)$ are independent Gaussian white noises with correlation functions

$$E[\xi_x(t)\xi_y(t')] = 0, \quad E[\xi_x(t)\xi_x(t')] = 2D\delta(t-t'),$$

$$E[\xi_y(t)\xi_y(t')] = 2D\delta(t-t'), \tag{38}$$

where $2D$ is the strength of the Gaussian white noises. In the case of thermal equilibrium, we may assume that the fluctuation-dissipation theorem (Einstein relation) applies: $D=\gamma_0 k_B T/m$, where T is the temperature and k_B is the Boltzmann constant.

Generally, the friction coefficient F is a function of the system state $(x,y,\dot{x},\dot{y})$. In this paper, velocity-dependent friction models $F=F(\sqrt{\dot{x}^2+\dot{y}^2})$, such as the Rayleigh, Schienbein-Gruler, and Erdmann velocity-dependent friction models are considered.

The Rayleigh velocity-dependent friction model is of the form [29]

$$F = -\gamma_1 + \gamma_2(\dot{x}^2 + \dot{y}^2). \tag{39}$$

For the case of $\gamma_1, \gamma_2 > 0$, the passive motion of Brownian particles could be changed into active motion. In the range of small velocity, i.e., $(\sqrt{\dot{x}^2+\dot{y}^2} < \sqrt{\gamma_1/\gamma_2})$, pumping due to negative friction occurs as an additional source of energy. Hence, slow particles are accelerated. On the other hand, the motion of fast particles, i.e., $(\sqrt{\dot{x}^2+\dot{y}^2} > \sqrt{\gamma_1/\gamma_2})$, is damped due to positive friction.

In the following it will be shown that using the stochastic averaging method for quasi-integrable Hamiltonian systems, one can obtain an analytical stationary solution for the Fokker-Planck equation associated with the system (37).

Because of the unit mass $m=1$, the generalized displacements q_1,q_2 and generalized momenta p_1,p_2 in system (37) are the same as x,y and $\dot{x},\dot{y}$, respectively. Following the derivation from Eqs. (3a) and (3b) to Eqs. (6a) and (6b), the following Itô stochastic differential equations can be obtained from system (37):

$$dx = \dot{x}dt, \quad d\dot{x} = -\{\omega^2 x + [-\gamma_1 + \gamma_2(\dot{x}^2 + \dot{y}^2)]\dot{x}\}dt + DdB_x(t),$$

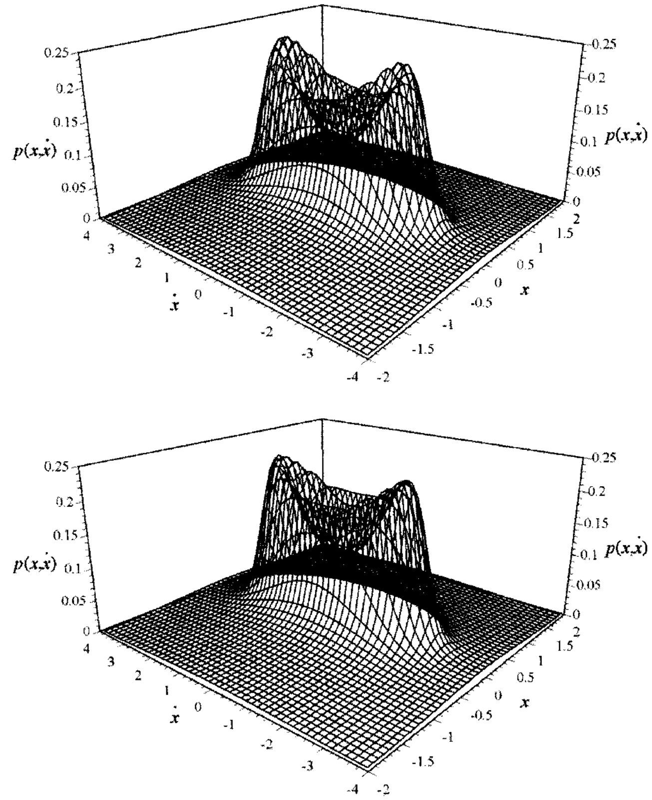

FIG. 3. Stationary probability density $p(x,\dot{x})$ of displacement x and velocity $\dot{x}$ of system (37) with Rayleigh friction model in the case far from equilibrium (top) from simulation of Eq. (37) and (bottom) from the analytical expression in Eq. (55). The parameters are the same as those in Fig. 1.

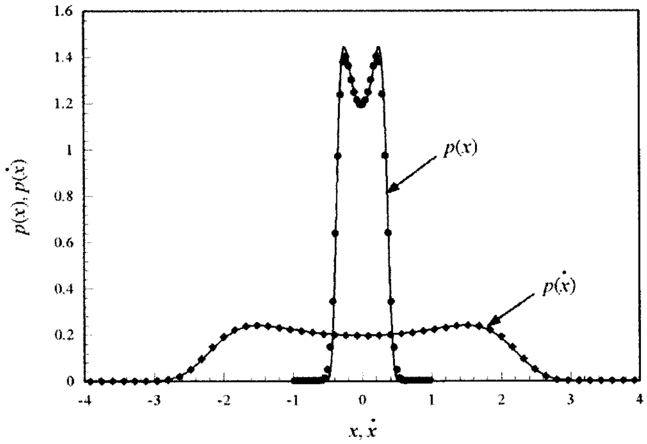

FIG. 4. Stationary probability densities $p(x)$ of displacement x and $p(\dot{x})$ of velocity $\dot{x}$ of system (37) with Rayleigh friction model in the case far from equilibrium. The parameters are the same as those in Fig. 1. ●, ◆ denote the results from simulation of Eq. (37) and — the analytical results in Eq. (55).

$$dy=\dot{y}dt,\quad d\dot{y}=-\{\omega^2 y+[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{y}\}dt+DdB_y(t), \tag{40}$$

where $B_x(t)$ and $B_y(t)$ are the standard Wiener processes. The Hamiltonian system associated with system (37) is integrable with two independent integrals of motion H_x and H_y, i.e.,

$$H_x=\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2,\quad H_y=\frac{1}{2}\dot{y}^2+\frac{1}{2}\omega^2y^2, \tag{41}$$

which are in involution. Any other constant of motion can be expressed in terms of these two motion integrals. For example, the total energy or Hamiltonian H and the angular momentum L of system (37) can be expressed as follows:

$$H=H_x+H_y,\quad L=\dot{x}y-y\dot{x}=k\sqrt{H_xH_y}, \tag{42}$$

where k is a constant determined by the initial state of the system.

Note that the natural frequencies in the x and y directions are the same and a resonant relation (16) with $k_1'=1$, $k_2'=-1$, and $0(\varepsilon)=0$ exists. So, in the case of light friction and weak stochastic excitations, the stochastic averaging method for quasi-integrable Hamiltonian systems with internal resonance introduced in Sec. III should be applied to Eq. (40). Introduce the angle variables θ_x and θ_y,

$$\theta_x=\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right),\quad \theta_y=\tan^{-1}\left(\frac{\dot{y}}{\omega y}\right). \tag{43}$$

According to Eqs. (8a) and (8b), the following Itô stochastic differential equations for H_x, H_y, θ_x, and θ_y can be obtained from Eq. (40):

$$dH_x=\{-[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{x}^2+D\}dt+\sqrt{2D}\dot{x}dB_x(t),$$

$$dH_y=\{-[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{y}^2+D\}dt+\sqrt{2D}\dot{y}dB_y(t),$$

$$d\theta_x=\left\{\omega-\frac{\omega[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]x\dot{x}}{\dot{x}^2+\omega^2x^2}+D\frac{2\omega x\dot{x}}{(\dot{x}^2+\omega^2x^2)^2}\right\}dt+\frac{\omega x}{\dot{x}^2+\omega^2x^2}dB_x(t),$$

$$d\theta_y=\left\{\omega-\frac{\omega[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]y\dot{y}}{\dot{y}^2+\omega^2y^2}+D\frac{2\omega y\dot{y}}{(\dot{y}^2+\omega^2y^2)^2}\right\}dt+\frac{\omega y}{\dot{y}^2+\omega^2y^2}dB_y(t). \tag{44}$$

Let $\phi=\theta_y-\theta_x$. Following Eq. (19), the averaged Fokker-Planck equation for Eq. (40) is

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial H_x}[a_xp]-\frac{\partial}{\partial H_y}[a_yp]-\frac{\partial}{\partial\phi}[a_\phi p]+\frac{1}{2}\frac{\partial^2}{\partial H_x^2}[b_{xx}p]+\frac{1}{2}\frac{\partial^2}{\partial H_y^2}[b_{yy}p]+\frac{1}{2}\frac{\partial^2}{\partial\phi^2}[b_{\phi\phi}p]+\frac{1}{2}\frac{\partial^2}{\partial H_x\partial H_y}[b_{xy}p]+\frac{1}{2}\frac{\partial^2}{\partial H_x\partial\phi}[b_{x\phi}p]+\frac{1}{2}\frac{\partial^2}{\partial H_y\partial H_x}[b_{yx}p]+\frac{1}{2}\frac{\partial^2}{\partial H_y\partial\phi}[b_{y\phi}p]+\frac{1}{2}\frac{\partial^2}{\partial\phi\partial H_x}[b_{\phi x}p]$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \phi\, \partial H_y}[b_{\phi y}p], \tag{45}$$

where

$$a_x = \langle -[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{x}^2 + D\rangle, \quad b_{xx}=\langle 2D\dot{x}^2\rangle,$$

$$b_{xy}=b_{x\phi}=0,$$

$$a_y = \langle -[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{y}^2 + D\rangle, \quad b_{yy}=\langle 2D\dot{y}^2\rangle,$$

$$b_{yx}=b_{y\phi}=0,$$

$$a_\phi = \left\langle -[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\left(\frac{\omega y\dot{y}}{\dot{y}^2+\omega^2y^2}-\frac{\omega x\dot{x}}{\dot{x}^2+\omega^2x^2}\right) + 2D\left[\frac{\omega y\dot{y}}{(\dot{y}^2+\omega^2y^2)^2}-\frac{\omega x\dot{x}}{(\dot{x}^2+\omega^2x^2)^2}\right]\right\rangle,$$

$$b_{\phi\phi}=\left\langle\left[\frac{2D\omega^2x^2}{(\dot{x}^2+\omega^2x^2)^2}+\frac{2D\omega^2y^2}{(\dot{y}^2+\omega^2y^2)^2}\right]\right\rangle,$$

$$b_{\phi x}=b_{\phi y}=0, \quad \langle\cdot\rangle = \frac{1}{2\pi}\int_0^{2\pi}(\cdot)d\theta_x. \tag{46}$$

To complete the averaging operation in Eq. (46), introduce the following transformation:

$$x=\frac{\sqrt{2H_x}}{\omega}\cos\,\theta_x, \quad \dot{x}=\sqrt{2H_x}\sin\,\theta_x,$$

$$y=\frac{\sqrt{2H_y}}{\omega}\cos\,(\phi+\theta_x), \quad \dot{y}=\sqrt{2H_y}\sin\,(\phi+\theta_x), \tag{47}$$

which satisfy Eq. (41). The exact stationary solution of the averaged Fokker-Planck equation (45), if it exists, is of the form

$$p(H_x,H_y,\phi)=C\exp[-\lambda(H_x,H_y,\phi)], \tag{48}$$

where C is a normalization constant and the probability potential $\lambda(H_x,H_y,\phi)$ is the solution of the following three linear partial differential equations:

$$b_{xx}\frac{\partial\lambda}{\partial H_x}=2D-2a_x, \quad b_{yy}\frac{\partial\lambda}{\partial H_y}=2D-2a_y,$$

$$b_{\phi\phi}\frac{\partial\lambda}{\partial\phi}=-2a_\phi. \tag{49}$$

Equation (49) be simplified as follows:

$$\frac{\partial\lambda}{\partial H_x}=\frac{1}{2\pi DH_x}\int_0^{2\pi}[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{x}^2d\theta_x,$$

$$\frac{\partial\lambda}{\partial H_y}=\frac{1}{2\pi DH_y}\int_0^{2\pi}[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\dot{y}^2d\theta_x,$$

$$\frac{\partial\lambda}{\partial\phi}=\frac{\omega H_xH_y}{\pi D(H_x+H_y)}\int_0^{2\pi}[-\gamma_1+\gamma_2(\dot{x}^2+\dot{y}^2)]\left(\frac{y\dot{y}}{H_y}-\frac{x\dot{x}}{H_x}\right)d\theta_x. \tag{50}$$

Finishing the integrals in Eq. (50), one obtains the following expressions:

$$\frac{\partial\lambda}{\partial H_x}=\frac{1}{D}\left[-\gamma_1+\frac{3}{2}\gamma_2H_x+\gamma_2\left(1+\frac{1}{2}\cos\,2\phi\right)H_y\right],$$

$$\frac{\partial\lambda}{\partial H_y}=\frac{1}{D}\left[-\gamma_1+\frac{3}{2}\gamma_2H_y+\gamma_2\left(1+\frac{1}{2}\cos\,2\phi\right)H_x\right],$$

$$\frac{\partial\lambda}{\partial\phi}=-\frac{1}{D}\gamma_2H_xH_y\sin\,2\phi. \tag{51}$$

To have an exact solution for λ, the following compatibility conditions must be satisfied:

$$\frac{\partial^2\lambda}{\partial H_x\partial H_y}=\frac{\partial^2\lambda}{\partial H_y\partial H_x}, \quad \frac{\partial^2\lambda}{\partial H_x\partial\phi}=\frac{\partial^2\lambda}{\partial\phi\partial H_x},$$

$$\frac{\partial^2\lambda}{\partial\phi\partial H_y}=\frac{\partial^2\lambda}{\partial H_y\partial\phi}. \tag{52}$$

It is seen that the compatibility conditions in Eq. (52) are satisfied spontaneously. Thus, the probability potential λ of Eq. (48) is

$$\lambda(H_x,H_y,\phi)=\frac{1}{D}\left\{-\gamma_1(H_x+H_y)+\gamma_2\left[\frac{3}{4}(H_x^2+H_y^2)+\left(1+\frac{1}{2}\cos\,2\phi\right)H_xH_y\right]\right\}. \tag{53}$$

The stationary probability density $p(H_x,H_y,\phi)$ is obtained by substituting Eq. (53) into Eq. (48) and the stationary probability density of displacements and velocities, following Eq. (36), is

$$p(x,y,\dot{x},\dot{y})=C_2\exp[-\lambda(H_x,H_y,\phi)]_{H_x=(\dot{x}^2+\omega^2x^2)/2,H_y=(\dot{y}^2+\omega^2y^2)/2,\phi=\tan^{-1}(\dot{y}/\omega y)-tg^{-1}(\dot{x}/\omega x)}, \tag{54}$$

where C_2 is a normalization constant. The other statistics of the stationary motion of the system (37) can then be obtained from Eq. (54). For example, the marginal stationary probability densities $p(x,y)$, $p(\dot{x},\dot{y})$, $p(x,\dot{x})$, $p(x)$, and $p(\dot{x})$ can be obtained as follows:

$$p(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,\dot{x},\dot{y})d\dot{x}\, d\dot{y},$$

$$p(\dot{x},\dot{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,\dot{x},\dot{y})dx\, dy,$$

$$p(x,\dot{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,\dot{x},\dot{y})dy\, d\dot{y},$$

$$p(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,\dot{x},\dot{y})d\dot{x}\, dy\, d\dot{y},$$

$$p(\dot{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,\dot{x},\dot{y})dx\, dy\, d\dot{y}. \tag{55}$$

To check the accuracy of the results obtained by using the stochastic averaging method, Monte Carlo simulation of the Langevin equation (37) was performed. The sample functions for independent Gaussian white noises $\xi_x(t)$ and $\xi_y(t)$ were generated by using the Box-Muller method. Then, the response of the system (37) was solved numerically by using the fourth-order Runge-Kutta method with time step 0.02. The long time solution after 60 000 steps was regarded as the stationary ergodic response and taken to perform the statistical analysis for obtaining the probability densities.

Figures 1–5 show some numerical results for five marginal probability densities of the system (37) in the case of far from equilibrium ($\gamma_2=1, \gamma_1/\gamma_2=4$) and in the case of tending toward equilibrium ($\gamma_2=1, \gamma_1/\gamma_2=-1$). The parameters are the same as those used in Ref. [30]. The validity of the stationary solution (54) is verified by the excellent agreement between the results from the stochastic averaging method and those from digital simulation for both cases. It is interesting to point out that the phenomenological bifurcation of system (37) can be studied analytically by using Eqs. (54) and (55).

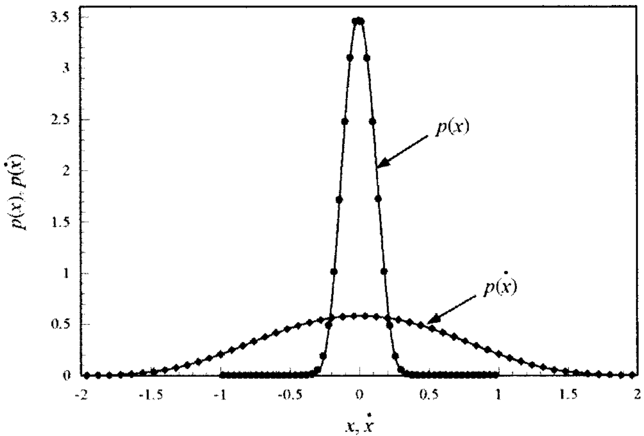

FIG. 5. Stationary probability densities $p(x)$ of displacement x and $p(\dot{x})$ of velocity $\dot{x}$ of system (37) with Rayleigh friction model in the case tending toward equilibrium. The parameters are $\gamma_1=-1$, $\gamma_2=1$, $\omega=6$, and $D=1$. ●, ♦ denote the results from simulation of Eq. (37) and — the analytical results in Eq. (55).

VI. STATIONARY SOLUTIONS FOR SCHIENBEIN-GRULER AND ERDMANN FRICTION MODELS

The stochastic averaging method for quasi-integrable Hamiltonian systems is also applicable to the system (37) with other velocity-dependent friction models. Replacing the Rayleigh friction model with the following Schienbein-Gruler velocity-dependent friction model [1]:

$$F(\sqrt{\dot{x}^2+\dot{y}^2}) = \gamma_0\left(1-\frac{v_0}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) \tag{56}$$

which describes the active motion of different types of cell, such as granulocytes, monocytes, and neural crest cells, and by using the stochastic averaging method for quasi-integrable Hamiltonian systems, one can obtain the following joint stationary probability density of system (37):

$$p(x,y,\dot{x},\dot{y}) = C\exp\left[-\frac{\gamma_0}{D}(H_x+H_y)+\frac{\sqrt{2}v_0\gamma_0}{\pi D}\times\int_0^{2\pi}\sqrt{H_x\sin^2\theta+H_y\sin^2(\theta+\phi)}d\theta\right]\Bigg|_{H_x=(\dot{x}^2+\omega^2x^2)/2,H_y=(\dot{y}^2+\omega^2y^2)/2,\phi=\tan^{-1}(\dot{y}/\omega y)-tg^{-1}(\dot{x}/\omega x)}. \tag{57}$$

Similarly, for system (37) with the Erdmann velocity-dependent friction model [31]

$$F(\sqrt{\dot{x}^2+\dot{y}^2}) = \gamma_0\frac{(\dot{x}^2+\dot{y}^2-v_0^2)}{(q_0/\gamma_0)+(\dot{x}^2+\dot{y}^2-v_0^2)} \tag{58}$$

the following joint stationary probability density of the system (37) can be obtained:

$$p(x,y,\dot{x},\dot{y}) = C \exp\left\{ -\frac{\gamma_0}{D}(H_x + H_y) + \frac{q_0}{2\pi D} \times \int_0^{2\pi} \ln[q_0/\gamma_0 + 2H_x \sin^2\theta + 2H_y \sin^2(\theta+\phi) - v_0^2] d\theta \right\} \Bigg|_{H_x=(\dot{x}^2+\omega^2x^2)/2, H_y=(\dot{y}^2+\omega^2y^2)/2, \phi=\tan^{-1}(\dot{y}/\omega y)-tg^{-1}(\dot{x}/\omega x)} . \tag{59}$$

It is noted that similar results as shown in Figs. 1–5 can be obtained for the system (37) with Schienbein-Gruler and Erdmann friction models.

VII. CONCLUSIONS

In the present paper, the stochastic averaging method for quasi-integrable Hamiltonian systems has been introduced briefly and applied to obtain the stationary distribution for the motion of active Brownian particles. The method can be used to deal with a system with different kinds of friction models both in cases far from equilibrium and tending toward equilibrium. For three velocity-dependent friction models, i.e., the Rayleigh model, Schienbein-Gruler model, and Erdmann model, the analytical solutions obtained agree well with the results from Monte Carlo simulation. It should be pointed out that the stochastic averaging method for quasi-integrable Hamiltonian systems can also be extended to the study of the motion of swarms of active Brownian particles with global coupling and it will be the subject of our future study.

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